The weight, density and Lindelöf number in spaces and topological groups

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It is well known that every regular space $X$ satisfies $w(X) \leq 2^{d(X)}$, where $w(X)$ and $d(X)$ are the weight and density of $X$, respectively. For Tychonoff spaces, W. Comfort and A. Hager established a sharper result. They proved that the number of continuous real-valued functions on a Tychonoff space $X$, say $C(X)$, satisfies $w(X) \leq |C(X)| \leq w(X)^{\text{wl}(X)}$, where $\text{wl}(X)$ is the weak Lindelöf number of $X$. Notice that $w(X)^{\text{wl}(X)} \leq 2^{d(X)}$ for every regular space $X$.

We present new upper bounds for the weight of spaces and topological groups which contain a dense Lindelöf $\Sigma$-subspace. One of our principal results states that if $Y$ is a dense subspace of a Tychonoff space $X$, then $w(X) \leq |C(X)| \leq nw(Y)^{\text{Nag}(Y)}$, where $\text{Nag}(Y)$ is the Nagami number of $Y$. In particular, if a regular Lindelöf $\Sigma$-space $X$ satisfies $nw(X) = \kappa^\omega$ for some $\kappa \geq \omega$, then $w(X) = nw(X)$ and $w(\beta X) = |C(X)| = \kappa^\omega$. Therefore, the cardinality of $C(X)$ is completely defined by the weight of $X$ in this case.

The upper bounds for the weight of topological groups are even better. We show that if a Lindelöf $\Sigma$-group $G$ is a dense subgroup of a topological group $H$, then $w(H) = w(G) \leq \psi(G)^\omega$. Similarly, if a Lindelöf $\Sigma$-space $X$ generates a dense subgroup of a topological group $H$, then $w(H) \leq 2^{\psi(X)}$. 