Computational Methods and Function Theory
https://doi.org/10.1007/s40315-022-00470-4

Lawrence Allen Zalcman 1943–2022

Mark Agranovsky[1] · **Walter Bergweiler**[2]

Received: 26 July 2022 / Revised: 2 August 2022 / Accepted: 2 August 2022

Photo courtesy of Bildarchiv des Mathematischen Forschungsinstituts Oberwolfach

Lawrence Allen Zalcman was born on June 9, 1943, in Kansas City, Missouri (USA). Larry attended Dartmouth College, graduating from there in 1964. Afterwards he went to MIT, where he received his Ph.D. under the supervision of Kenneth Hoffman in 1968. He then became Assistant Professor at Stanford University. In 1970–1971 he took a leave from Stanford to visit the Hebrew University in Jerusalem. In 1972 he became Associate Professor at the University of Maryland, being promoted to the rank of full professor in 1974. Further visits to Israel followed, to the Weizmann Institute, the Technion, the Hebrew University and Bar-Ilan University. Overall the Zalcman family spent about one third of the period 1970–1981 in Israel. Eventually, in 1985, Bar-Ilan University made him an "offer he couldn't refuse". Larry became the Lady Davis Chair in Mathematics at Bar-Ilan University and moved with his family to Israel

✉ Walter Bergweiler
bergweiler@math.uni-kiel.de

Mark Agranovsky
agranovs@math.biu.ac.il

[1] Bar-Ilan University, Ramat Gan 52900, Israel

[2] Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, 24098 Kiel, Germany

Published online: 28 October 2022

for good. He remained at Bar-Ilan University ever since. Larry passed away on May 31, 2022, in Jerusalem.

Let us turn to some of Larry's mathematical achievements. Naturally, there will be some overlap with the previous accounts [AgSh, Dur] of his work. (Citations using abbreviated author names refer to the list of references at the end of this article, while references using numbers refer to bibliography of Larry's articles and books (https://link.springer.com/article/10.1007/s40315-022-00471-3).

Larry wrote his first mathematical paper [1] already at Dartmouth. This article, written jointly with Michael Voichik, was based on his undergraduate thesis.

At MIT, Larry joined a circle of talented young mathematicians that had grouped around his advisor Kenneth Hoffman. When he left MIT in 1968, he had already published six papers, as well as the monograph *Analytic capacity and Rational Approximation* [1], which immediately attracted international attention.

Larry's doctoral dissertation [7] was devoted to Banach algebras of bounded analytic functions on infinitely-connected domains in the complex plane. At the same time, his mathematical interests also included harmonic analysis, related to convolution equations.

While he was still at Stanford, he proved a striking Morera-type theorem [11]. Recall that if $f \in C(\mathbb{R}^2)$ satisfies

$$\int_C f(z)dz = 0$$

for every closed curve C, then f is an entire holomorphic function by Morera's theorem. It is easy to see that it suffices to restrict to circles C here, but it is not enough to restrict to circles of a fixed radius. Does it suffice to consider only circles of two fixed radii r_1 and r_2? Larry proved the surprising result that the answer is "yes" if and only if r_1/r_2 is not a quotient of zeros of the Bessel function J_1.

This theorem, known today as Zalcman's two circles theorem, triggered a new direction in the study of convolution equations, called "two radii theorems". In the excellent paper [12], Larry developed his idea and generalized it to arbitrary PDEs with constant coefficients. It was shown there that the weak solutions of any differential equation of the form $P(D)f = 0$, where P is a homogeneous polynomial in n variables and $D = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$, may be characterized by appropriate two radii theorems. The initial Morera-type two circles theorem corresponds to the case $P(x_1, x_2) = \frac{1}{2}(x_1 + ix_2)$. In the joint article with Carlos Berenstein [21] these results were extended to rank one symmetric spaces. The fascinating mix of function theory, PDE, and harmonic analysis and integral geometry on Euclidean and non-Euclidean spaces was summarized in the beautiful survey [20] in the American Mathematical Monthly. This highly inspiring article won the Lester R. Ford Award from the Mathematical Association of America and is an excellent sample of Larry's talent of mathematical writing. Analogs of the two circle theorem discovered by Larry are now known for Lie groups, in particular Heisenberg groups [ABCP], and other homogeneous spaces. The study of local versions of this theorem led to deep questions of spectral synthesis and number theory (see, e.g., the book [VoVo]). The field opened by Zalcman's two radii theorem remains active today.

Another problem from PDE and harmonic analysis, related to convolution equations, which was within Larry's interests, is known as the Pompeiu problem. It can be formulated in different equivalent terms, for example as a uniqueness theorem of a convolution equation on the motion group of $\mathbb{R}^n$ and also as an existence of a solution for an inverse overdetermined problem with constant Dirichlet–Neumann boundary conditions. Larry's contribution to the study of this problem include not only the papers [11] and [21] already mentioned, but also the frequently cited bibliographic survey [32].

Talking about inverse problems, it is impossible not to mention another masterpiece of Larry's (with Dov Aharonov and Max Schiffer), the article *Potato Kugel* [22] where the authors proved that a homogeneous solid ("potato") exerting a gravitational attraction identical to that exerted by a point mass must be a ball ("Kugel"). Larry liked to tell the story that when he gave a talk on this at the Colloquium at Hebrew University, he arranged to have the usual cookies and tea before the talk be replaced by generous servings of potato kugel. The colloquium—or at least the tea—was the best attended ever. Larry enjoyed good food and used culinary terms in mathematical texts, for example, *Potato Kugel* in [22] was followed by a section in [28] entitled *Matzoh Ball Soup*. About the impact of the latter paper we only mention that MathSciNet now lists four papers having "Matzoh ball soup" in the title.

In 1975, Larry turned his attention also to a very different problem, which would occupy him for many years to come. This problem concerns a heuristic principle attributed to André Bloch which asserts that "a family of holomorphic (meromorphic) functions which have a property P in common in a domain D is a normal family in D if P cannot be possessed by non-constant entire (meromorphic) functions in the finite plane" [Hil, p. 250]. In his retiring presidential address to the Association for Symbolic Logic, Abraham Robinson [Rob] listed the explication of this principle as one of twelve problems worthy of the attention of mathematicians. In [15], Larry gave such an explication for many important instances of Bloch's Principle. The key tool was not—as Robinson might have hoped—non-standard analysis, but a fairly elementary lemma, now known as Zalcman's Lemma. It says that a family $\mathcal{F}$ of functions meromorphic in a domain D is not normal if and only if there exists a sequence (z_n) in D converging to a point in D, a sequence (ρ_n) of positive real numbers tending to 0, a sequence (f_n) in $\mathcal{F}$ and a non-constant function g meromorphic in $\mathbb{C}$ such that

$$g_n(z) := f_n(z_n + \rho_n z) \to g(z)$$

locally uniformly in $\mathbb{C}$. While the statement is somewhat technical, the underlying idea is simple: For suitable properties P the limit function g will inherit the property P from the functions f_n, thereby verifying Bloch's principle for such a property. An important example is the property to omit three given values, yielding an easy deduction of Montel's Theorem from Picard's Theorem.

Xuecheng Pang [Pan] increased the applicability of Zalcman's Lemma considerably by showing that instead of the functions g_n given above one may also take $g_n(z) := \rho_n^\alpha f_n(z_n + \rho_n z)$ for certain $\alpha \in \mathbb{R}$. This is particularly useful when dealing with properties involving derivatives. The definite result of this type, yielding the maximal

range of α under suitable hypotheses, is given in a joint paper by Xuecheng Pang and Larry Zalcman [40].

Surprisingly, Zalcman's Lemma allows not only to show that certain results about normal families and functions meromorphic in the plane are *equivalent*, as predicted by Bloch's Principle, it actually yields *proofs of these results*. In particular, Zalcman's Lemma leads to a very short proof of Picard's and hence Montel's Theorem. This beautiful proof (due to Antonio Ros) is contained in Larry's excellent survey [36] of the subject. Zalcman's Lemma also yields a fairly short proof of the Ahlfors Five Islands Theorem [Ber].

Among later applications we mention a result of Larry from the joint paper [58] with Xuecheng Pang and his former student Shahar Nevo saying that if a transcendental meromorphic function has only finitely many simple zeros, then its derivative takes every non-zero value infinitely often. For some applications of Zalcman's Lemma it is important that the limit function g occurring there has finite order. This allows to prove results for all meromorphic functions by reducing them to the case of finite order functions. This argument appears first in an unpublished manuscript [1] of Larry and in [BeEr, ChFa]; see [36, p. 226] for a discussion. It has subsequently been used in various other papers.

Zalcman's Lemma has been generalized in various directions—and the explication of Bloch's Principle correspondingly extended. For example, there is an extension to quasiregular maps in higher dimensions by Ruth Miniowitz [Min]. This has been used by Alexandre Eremenko [Ere] to extend Bloch's theorem to quasiregular mappings and by Mario Bonk and Juha Heinonen [BoHe] in their work on *Quasiregular mappings and cohomology*.

As another instance where the ideas of Zalcman's lemma have been successfull we mention the work of Robert Brody [Bro] on *Compact manifolds and hyperbolicity*; cf. [Wu, p. 95].

The above is only a small selection of Larry's contribution to mathematics. The American Mathematical Society recognized his outstanding contributions by electing him to its Inaugural Class of Fellows in 2012.

The recognition of his work is also witnessed by the fact that a number of mathematical results and ideas are named after him. Zalcman's two-radii theorem and Zalcman's Lemma were explained above. There are also Zalcman domains [HKN] (which play a role in the classification of Riemann surfaces), Zalcman functions [Ste] (which arise in complex dynamics), and the Zalcman conjecture about univalent functions.

We have already mentioned Larry's brilliant surveys *Normal families: new perspectives* [36] and *Offbeat integral geometry* [20], and that he received the Lester R. Ford Award for expository excellence for the latter one in 1980. In fact, Larry's skills as an expositor, both on paper and in person, are legendary. His papers *Real proofs of complex theorems (and vice versa)* [13] and *A Tale of Three Theorems* [74] were also honored by a Lester R. Ford Award in 1975 and 2017, respectively. The first of these two papers also received the prestigious Chauvenet Prize from the Mathematical Association of America. As another masterpiece of exposition we mention Larry's article *Modern perspectives on classical function theory* [23]. Peter Lappan concludes his review [Lap] in *Mathematical Reviews* by saying that this article "should be required reading for all students in function theory courses". This could also be said about

his book, with Peter Lax, on *Complex proofs of real theorems* [2]. Note that the title echoes that of Larry's paper [13] mentioned above.

During all this time, Larry's service to the profession was no less distinguished than his mathematics. He was a member of the Council of the AMS (1979–1982) and chaired a number of important committees of the Mathematical Association of America. As a Chairman of the Joint AMS-ASL-IMS Committee on Translations, Larry successfully struggled to secure translation and publication rights for articles and books by mathematicians from the Soviet Union who were in official disfavor, whether because of their Jewish origin, dissidents politics, or having emigrated to the West. From 1997 to 1999, Zalcman served as President of the Israel Mathematical Union.

Larry was a member of the Editorial Board of the Proceedings of the American Mathematical Society from 1976 to 1982 and of Computational Methods and Function Theory from its first issue in 2001 until his death. But his most significant editorial work was certainly for Journal d'Analyse Mathématique, where he acted as Editor-in-Chief of from 1987 to 2017. Larry used his professionalism and editor's talent to enhance its prestige to one of the world's premier journals in analysis.

Given his international standing, it is not surprising that Larry also had a leading role in his own university. After becoming Professor at Bar-Ilan University, he founded there the Analysis Seminar which within a few years turned to one of the strongest and attractive centers of mathematical life in Israel. Larry brought literally dozens of distinguished analysts from all over the world to Israel and to Bar-Ilan as month-long visitors. Larry was the heart and motor of the Analysis Seminar. The Analysis Seminar, during Larry's heading it, was a real weekly mathematical fest and its atmosphere is unforgettable for those who were lucky to participate in Larry's seminar. Throughout his career, Larry has always made a point of fostering mathematical talent whenever he encountered it; and more than a few well-known mathematicians have benefited from his encouragement, advice, and personal support at an early stage in their careers. Larry made an invaluable contribution in the absorbtion of immigrant mathematicians from the former Soviet Union. His efforts were unstinting and largely successful; one result of these was the creation of the exceptionally strong group in analysis at Bar-Ilan, whose universally recognized leader was Larry.

Larry Zalcman's erudition and intellectual horizons were unusually broad, and his expertise in subjects which interested him unusually deep. He has published articles in leading academic journals in areas such as philosophy, history of science, literature, and rabbinics. Special mention should be made of Larry's interest in textual criticism of the Hebrew Bible. He has written a dozen of articles in this last area (including two encyclopedia entries) and his suggestions can be found with approval in standard academic biblical commentaries. Larry maintained a long-term and lively interest to cinema; in fact, as a graduate student, he almost became the film critic for *Boston After Dark*. We think that everybody, who was familiar with Larry, can recollect when he or she learned from conversation with Larry most interesting things from an unexpected area. Larry was a "walking encyclopedia" and many of us benefited from the broad scope of Larry's knowledge. Larry's special quality was his humor. Humor has always played an important role in Larry's approach to life in general and mathematics in particular.

Larry was very happy in his family life. He met his charming future wife Adrienne on his twenty-first birthday and used to call her "my best birthday present I ever got". They got married in 1965, and their two children were born in 1970 (Joel) and 1971 (Shuli). Larry also liked to say that for him, who has written more than thirty papers on normal families, it seems especially appropriate that his own family (besides his loving wife Adrienne, son Joel and daughter Shuli there are six grandchildren) is a very normal and happy one.

Larry was a true and reliable friend. He had many friends and it seemed sometimes that there is a Larry's friend in almost every country of the globe. Many of those who worked at Bar-Ilan University can tell stories about the reaction of mathematicians abroad to one's affiliation: "Bar-Ilan? That's where Larry Zalcman works?"

Larry often said that he considers himself "incredibly lucky" in both his personal and his professional life. Larry's friends and colleagues, and all who knew him well, are also incredibly lucky to have met Larry Zalcman on their life paths. They benefited a lot from Larry's extraordinary personality, from joy of doing or discussing mathematics with him, from his warm friendship and firm support.

We will be deeply missing you, Larry!

Funding Open Access funding enabled and organized by Projekt DEAL.

References

[ABCP] Agranovsky, M., Berenstein, C., Chang, D.-C., Pascuas, D.: Injectivity of the Pompeiu transform in the Heisenberg group. J. Anal. Math. **63**, 131–173 (1994)

[AgSh] Agranovsky, M., Shoikhet, D.: Lawrence Zalcman at sixty. Complex analysis and dynamical systems II, 1–6, Contemp. Math., 382, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI (2005)

[Ber] Bergweiler, W.: A new proof of the Ahlfors five islands theorem. J. Anal. Math. **76**, 337–347 (1998)

[BeEr] Bergweiler, W., Eremenko, A.: On the singularities of the inverse to a meromorphic function of finite order. Rev. Mat. Iberoamericana **11**(2), 355–373 (1995)

[BoHe] Bonk, M., Heinonen, J.: Quasiregular mappings and cohomology. Acta Math. **186**(2), 219–238 (2001)

[Bro] Brody, R.: Compact manifolds and hyperbolicity. Trans. Am. Math. Soc. **235**, 213–219 (1978)

[ChFa] Chen, H.H., Fang, M.L.: The value distribution of $f^n f'$. Sci. China Ser. A **38**(7), 789–798 (1995)

[Dur] Duren, P.: Lawrence Zalcman: a mathematical biosketch. Anal. Math. Phys. **4**(1–2), 1–5 (2014)

[Ere] Eremenko, A.: Bloch radius, normal families and quasiregular mappings. Proc. Amer. Math. Soc. **128**(2), 557–560 (2000)

[HKN] Hayashi, M., Kobayashi, Y., Nakai, M.: A uniqueness theorem and the Myrberg phenomenon for a Zalcman domain. J. Anal. Math. **82**, 267–283 (2000)

[Hil] Hille, E.: Analytic Function Theory, vol. 2. Ginn and Company, Boston-New York-Toronto (1962)

[Lap] Lappan, P.: Review of [23], MR 83d:30003

[Min] Miniowitz, R.: Normal families of quasimeromorphic mappings. Proc. Amer. Math. Soc. **84**(1), 35–43 (1982)

[Pan] Xue Cheng Pang: Bloch's principle and normal criterion. Sci. China Ser. A **32**(7), 782–791 (1989)
[Rob] Robinson, A.: Metamathematical problems. J. Symbol. Logic **38**, 500–516 (1973)
[Ste] Steinmetz, N.: Zalcman functions and rational dynamics. N. Z. J. Math. **32**(1), 91–104 (2003)
[VoVo] Volchkov, Valery V., Volchkov, Vitaly V.: Offbeat integral geometry on symmetric spaces. Birkhäuser/Springer, Basel (2013)
[Wu] Wu, H.: Some theorems on projective hyperbolicity. J. Math. Soc. Jpn. **33**(1), 79–104 (1981)